Universal extension classes for algebraic supergroups

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Let k be a field (usually of characteristic $p > 2$).

Cohomological Finite Generation Question

Suppose A is something like a finite-dimensional Hopf algebra over k.

- Is the cohomology ring $\mathrm{H}^\bullet(A, k)$ a finitely-generated k-algebra?
- For each finite-dimensional left A-module M, is $\mathrm{Ext}_A^\bullet(M, M)$ finitely-generated as a left module over $\mathrm{H}^\bullet(A, k)$?

Some affirmative answers ...

- Golod (1959): group algebras of finite p-groups
- Venkov, Evens (1961): group algebras of finite groups
- Friedlander–Parshall (1986): restricted enveloping algebras

...and then their "surprisingly elusive" unification:

- Friedlander–Suslin (1997): finite group schemes over a field (equivalently, finite-dimensional cocommutative k-Hopf algebras)

Other CFG results in the past 20 years, including by MPSW, TvdK, ...

Relevant to this talk: CFG for restricted Lie superalgebras (D, 2013)

What's so super about super linear algebra?

Something is "super" if it has a compatible grading by $\mathbb{Z}/2\mathbb{Z}$.

- Superspaces $V = V_{\overline{0}} \oplus V_{\overline{1}}$, $W = W_{\overline{0}} \oplus W_{\overline{1}}$
- Induced gradings on tensor products, linear maps, etc.

$$(V \otimes W)_\ell = \bigoplus_{i+j=\ell} V_i \otimes W_j$$

$$\mathrm{Hom}_k(V,W)_\ell = \{f \in \mathrm{Hom}_k(V,W) : f(V_i) \subseteq W_{i+\ell}\}$$

- $V \otimes W \cong W \otimes V$ via the **supertwist** $v \otimes w \mapsto (-1)^{\overline{v}\cdot\overline{w}} w \otimes v$

Define (Hopf) superalgebras and notions of (super)commutativity and (super)cocommutativity in terms of the "usual diagrams," but using the supertwist map whenever graded objects pass one another.

Examples of Hopf superalgebras

- Ordinary Hopf algebras (considered as purely even superalgebras).
- The exterior algebra $\Lambda(V)$ of a vector space V is a commutative, cocommutative Hopf superalgebra, but not an ordinary Hopf algebra.
- Let G be a finite group and V be a finite-dimensional kG-module. Form the smash product algebra $\Lambda(V)\#kG$: $u, w \in \Lambda(V)$, $g, h \in G$,

$$(u \otimes g) \cdot (w \otimes h) := (u \wedge [g \cdot w]) \otimes gh$$

  This is a finite-dimensional cocommutative Hopf superalgebra, and over $\mathbb{C}$ all such algebras have this form (Kostant).

Examples (continued)

If $\mathfrak{g}$ is a (restricted) Lie superalgebra, then its (restricted) enveloping superalgebra is an example of a Hopf superalgebra.

Recall that a **Lie superalgebra** $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ is a superspace equipped with an even linear map $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that for all homogeneous elements $x, y, z \in \mathfrak{g}$ one has:

- $[x, y] = -(-1)^{\overline{x}\cdot\overline{y}}[y, x]$
- $[x, [y, z]] = [[x, y], z] + (-1)^{\overline{x}\cdot\overline{y}}[y, [x, z]]$
- $[x, x] = 0$ if $x \in \mathfrak{g}_{\overline{0}}$ and $p = 2$
- $[x, [x, x]] = 0$ if $x \in \mathfrak{g}_{\overline{1}}$ and $p = 3$

Say that $\mathfrak{g}$ is a **restricted Lie superalgebra** if $\mathfrak{g}_{\overline{0}}$ is an ordinary restricted Lie algebra and the action of $\mathfrak{g}_{\overline{0}}$ on $\mathfrak{g}_{\overline{1}}$ makes $\mathfrak{g}_{\overline{1}}$ into a restricted $\mathfrak{g}_{\overline{0}}$-module.

Classical correspondences:

affine group schemes $\leftrightarrow$ cocommutative Hopf algebras
finite group schemes $\leftrightarrow$ f.d. cocommutative Hopf algebras
height-one infinitesimal group schemes $\leftrightarrow$ f.d. restricted Lie algebras

Super correspondences:

affine supergroup schemes $\leftrightarrow$ cocommutative Hopf superalgebras
finite supergroup schemes $\leftrightarrow$ f.d. cocommutative Hopf superalgebras
height-one infinitesimal supergroup schemes $\leftrightarrow$ f.d. res. Lie superalgebras

Main Theorem (D, 2014)

Let G be a finite k-supergroup scheme (equivalently, a finite-dimensional cocommutative Hopf superalgebra). Then the cohomology ring $\mathrm{H}^\bullet(G, k)$ is a finitely-generated k-superalgebra.

Remark

If A is a Hopf superalgebra, then the Radford biproduct $A\#(\mathbb{Z}/2\mathbb{Z})$ is an ordinary Hopf algebra, and $\mathrm{H}^\bullet(A\#(\mathbb{Z}/2\mathbb{Z}), k) \cong \mathrm{H}^\bullet(A, k)_{\overline{0}}$.

It is thus possible to view the main theorem as a generalization of the FS result in multiple ways (to a strictly larger class of ordinary Hopf algebras, or to Hopf algebra objects in another symmetric monoidal category).

Key steps in Friedlander and Suslin's argument:

- Reduce to when G is an infinitesimal subgroup scheme of GL_n.
- Reduce to the existence of certain "universal extension classes"

$$e_r \in \operatorname{Ext}_{GL_n}^{2p^{r-1}}((k^n)^{(r)}, (k^n)^{(r)}), \quad r \geq 1.$$

  These give rise by restriction to homomorphisms

$$e_r|_G : S^\bullet(\mathfrak{gl}_n^\#)^{(r)} \to \mathrm{H}^{2p^{r-1}\bullet}(G, k)$$

  whose images provide the algebra generators for $\mathrm{H}^\bullet(G, k)$.
- For $n \geq p^r$, use the isomorphism

$$\operatorname{Ext}_{GL_n}^\bullet((k^n)^{(r)}, (k^n)^{(r)}) \cong \operatorname{Ext}_{\mathcal{P}_{p^r}}^\bullet(I^{(r)}, I^{(r)}),$$

  to reduce to calculations in the category $\mathcal{P}_{p^r}$ of homogeneous strict polynomial functors of degree p^r (equivalent for $n \geq p^r$ to the cat. of degree-p^r homogeneous polynomial representations of GL_n).

Similar program for finite supergroup schemes:

- Reduce to when G is an infinitesimal subgroup scheme of the general linear supergroup scheme $GL(m|n)$.
- Reduce to the existence of certain "universal extension classes"

$$e_r^{m,n} \in \mathrm{Ext}_{GL(m|n)}^{2p^{r-1}}((k^{m|n})^{(r)}, (k^{m|n})^{(r)})$$
$$c_r^{m,n} \in \mathrm{Ext}_{GL(m|n)}^{p^r}((k^{0|n})^{(r)}, (k^{m|0})^{(r)})$$

  that give rise to certain algebra homomorphisms

$$e_r^{m,n}|_G : S^\bullet(\mathfrak{gl}(m|n)_0^\#)^{(r)} \to \mathrm{H}^{2p^{r-1}\bullet}(G,k)$$
$$c_r^{m,n}|_G : S^\bullet(\mathfrak{gl}(m|n)_1^\#)^{(r)} \to \mathrm{H}^{p^r\bullet}(G,k)$$

- **Work since 2013:** Exhibit the extension classes for $GL(m|n)$ by calculating extension groups in the category $\mathcal{P}$ of strict polynomial superfunctors defined by Axtell.

Example of an ordinary strict polynomial functor

Suppose V has basis $\{u, v\}$ and W has basis $\{x, y\}$.

Then $S^2(V)$ has basis $\left\{u^2, uv, v^2\right\}$ and $S^2(W)$ has basis $\left\{x^2, xy, y^2\right\}$.

Let $\phi : V \to W$ be the linear map with associated matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

The linear map $S^2(\phi) : S^2(V) \to S^2(W)$ is defined for $f \in S^2(V)$ by

$$S^2(\phi)(f(u, v)) = f(\phi(u), \phi(v)).$$

The associated matrix for $S^2(\phi)$ is then

$$\begin{pmatrix} a^2 & ab & b^2 \\ 2ac & (ad + cb) & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$

- $\boldsymbol{\mathcal{V}}$ category of finite-dimensional superspaces
- $V^{\otimes d}$ is naturally a right $\mathfrak{S}_d$-module (signed place permutations)
- $\boldsymbol{\Gamma}^d(V) = (V^{\otimes d})^{\mathfrak{S}_d}$
- $\boldsymbol{\Gamma}^d(\boldsymbol{\mathcal{V}})$: category with the same objects as $\boldsymbol{\mathcal{V}}$, but with morphisms

$$\mathrm{Hom}_{\boldsymbol{\Gamma}^d(\boldsymbol{\mathcal{V}})}(V, W) = \boldsymbol{\Gamma}^d[\mathrm{Hom}_k(V, W)] \cong \mathrm{Hom}_{k\mathfrak{S}_d}(V^{\otimes d}, W^{\otimes d}).$$

Strict polynomial superfunctors (Axtell, 2013)

The category $\boldsymbol{\mathcal{P}}_d$ of homogeneous degree-d strict polynomial superfunctors is the category of functors $F : \boldsymbol{\Gamma}^d\boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{V}}$ such that for each $V, W \in \boldsymbol{\mathcal{V}}$,

$$F_{V,W} : \mathrm{Hom}_{k\mathfrak{S}_d}(V^{\otimes d}, W^{\otimes d}) \to \mathrm{Hom}_k(F(V), F(W))$$

is an even k-linear map.

Examples of strict polynomial superfunctors

- Π parity flip functor $\quad (\Pi V)_{\overline{0}} = V_{\overline{1}},\ (\Pi V)_{\overline{1}} = V_{\overline{0}}$
- $\boldsymbol{T}^d(V) = V^{\otimes d}$ tensor power
- $\boldsymbol{\Gamma}^d(V) = (V^{\otimes d})^{\Sigma_d}$ super-symmetric tensors $\quad \boldsymbol{\Gamma}(V) = \Gamma(V_{\overline{0}}) \otimes \Lambda(V_{\overline{1}})$
- $\boldsymbol{S}^d(V) = (V^{\otimes d})_{\Sigma_d}$ super-symmetric power $\quad \boldsymbol{S}(V) = S(V_{\overline{0}}) \otimes \Lambda(V_{\overline{1}})$
- $\boldsymbol{\Lambda}^d(V)$ super-exterior power $\quad \boldsymbol{\Lambda}(V) = \Lambda(V_{\overline{0}}) \,{}^{g}\!\otimes S(V_{\overline{1}})$
- $\boldsymbol{A}^d(V)$ super-alternating tensors $\quad \boldsymbol{A}(V) = \Lambda(V_{\overline{0}}) \,{}^{g}\!\otimes \Gamma(V_{\overline{1}})$
- $\boldsymbol{I}^{(r)}(V) = V^{(r)}$ r-th Frobenius twist $(r \geq 1)$ $\quad \boldsymbol{I}^{(r)} = \boldsymbol{I}_0{}^{(r)} \oplus \boldsymbol{I}_1{}^{(r)}$
- Non-example: $V \mapsto V_{\overline{0}}$ (incompatible with composition of odd maps)

- SPSFs restrict to ordinary SPFs in two different ways
- Ordinary SPFs in general don't seem lift to SPSFs
- Frobenius twists of SPFs lift to SPSFs in several different ways

Goal: Calculate the structure of the extension algebra

$$\mathrm{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}^{(r)}, \boldsymbol{I}^{(r)}) = \begin{pmatrix} \mathrm{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}_0^{(r)}, \boldsymbol{I}_0^{(r)}) & \mathrm{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}_1^{(r)}, \boldsymbol{I}_0^{(r)}) \\ \mathrm{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}_0^{(r)}, \boldsymbol{I}_1^{(r)}) & \mathrm{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}_1^{(r)}, \boldsymbol{I}_1^{(r)}) \end{pmatrix}$$

Key tool: super analogue $\boldsymbol{\Omega} = \boldsymbol{S} \otimes \boldsymbol{A}$ of the de Rham complex.

$$\boldsymbol{\Omega}_n : \boldsymbol{S}^n \to \boldsymbol{S}^{n-1} \otimes \boldsymbol{A}^1 \to \boldsymbol{S}^{n-2} \otimes \boldsymbol{A}^2 \to \cdots \to \boldsymbol{S}^1 \otimes \boldsymbol{A}^{n-1} \to \boldsymbol{A}^n$$

Theorem ("super" Cartier isomorphism)

For each $n \in \mathbb{N}$, there exists an isomorphism of strict polynomial functors $\mathrm{H}^\bullet(\boldsymbol{\Omega}_{pn}) \cong \boldsymbol{\Omega}_n^{(1)}$, though it **does not** preserve the cohomological degree.

Use this to argue inductively via hypercohomology spectral sequences, though the failure to preserve the cohomological degree causes interesting things to happen during the base case of induction.

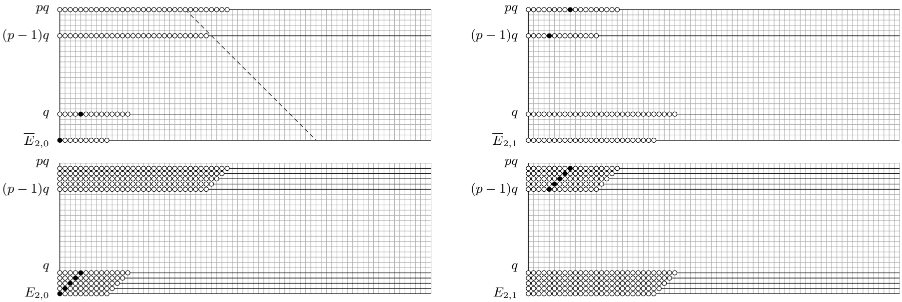
Theorem (D, 2014)

$\mathrm{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}^{(r)}, \boldsymbol{I}^{(r)})$ is generated as an algebra by extension classes

- $\boldsymbol{e}_i \in \mathrm{Ext}^{2p^{i-1}}_{\mathcal{P}}(\boldsymbol{I}_0{}^{(r)}, \boldsymbol{I}_0{}^{(r)})$ and $\boldsymbol{e}_i^\Pi \in \mathrm{Ext}^{2p^{i-1}}_{\mathcal{P}}(\boldsymbol{I}_1{}^{(r)}, \boldsymbol{I}_1{}^{(r)}) \quad (1 \leq i \leq r)$
- $\boldsymbol{c}_r \in \mathrm{Ext}^{p^r}_{\mathcal{P}}(\boldsymbol{I}_1{}^{(r)}, \boldsymbol{I}_0{}^{(r)})$ and $\boldsymbol{c}_r^\Pi \in \mathrm{Ext}^{p^r}_{\mathcal{P}}(\boldsymbol{I}_0{}^{(r)}, \boldsymbol{I}_1{}^{(r)})$

These generators satisfy:

- $(\boldsymbol{e}_i)^p = 0 = (\boldsymbol{e}_i^\Pi)^p$
- The $\boldsymbol{e}_i$ generate a commutative subalgebra, and similarly for the $\boldsymbol{e}_i^\Pi$
- The $\boldsymbol{e}_i$ restrict to Friedlander and Suslin's universal extension classes
- $(\boldsymbol{c}_r \cdot \boldsymbol{c}_r^\Pi)$ and $(\boldsymbol{c}_r^\Pi \cdot \boldsymbol{c}_r)$ generate polynomial subalgebras.
- Have $\boldsymbol{e}_i \cdot \boldsymbol{c}_r = \pm(\boldsymbol{c}_r \cdot \boldsymbol{e}_i{}^\Pi)$. But is it $+$ or $-$?
- $\boldsymbol{c}_r$ generates $\mathrm{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}_1{}^{(r)}, \boldsymbol{I}_0{}^{(r)})$ over the matrix ring
- $\boldsymbol{c}_r^\Pi$ generates $\mathrm{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}_0{}^{(r)}, \boldsymbol{I}_1{}^{(r)})$ over the matrix ring
- For $r \geq 1$, the $\boldsymbol{e}_r, \boldsymbol{c}_r, \boldsymbol{c}_r^\Pi$ restrict nontrivially to $GL(m|n)_1$.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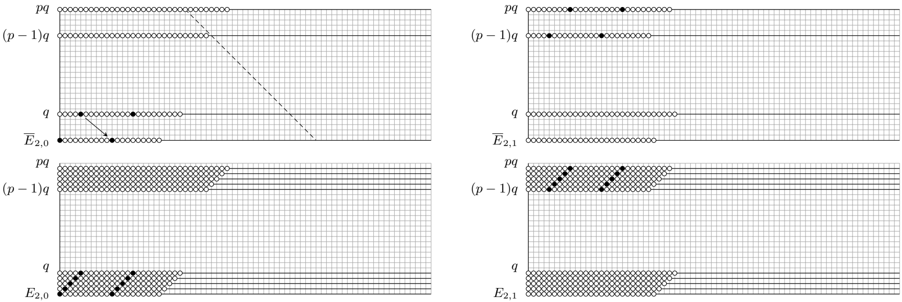
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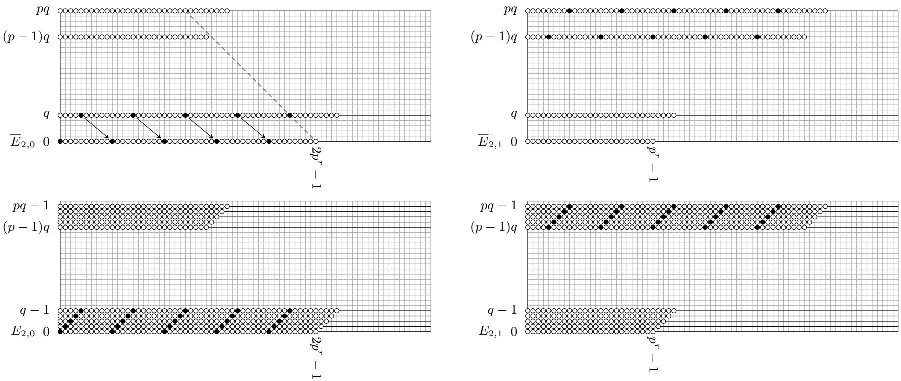

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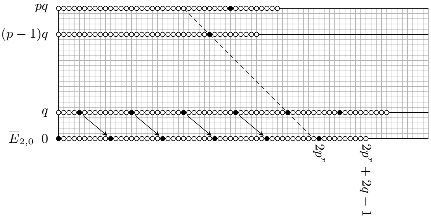
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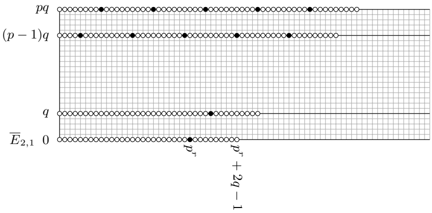
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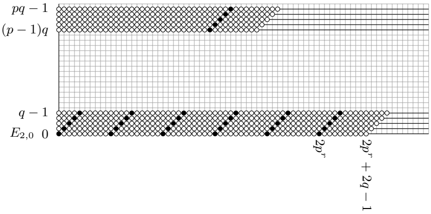
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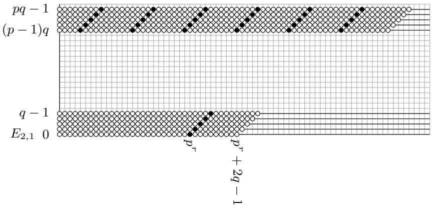
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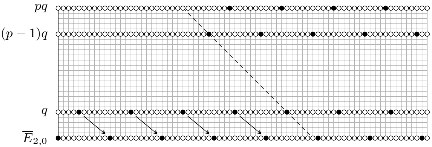
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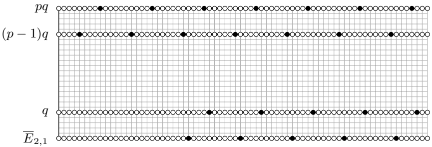
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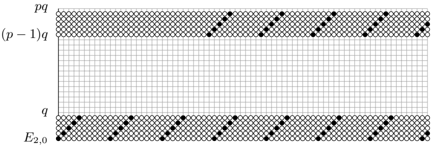
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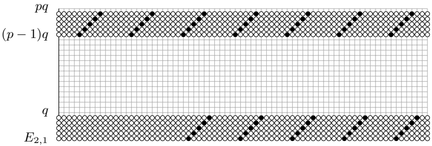
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Comments:

- The argument to determine the differentials in the spectral sequences, hence to determine the multiplicative structure in $\mathrm{Ext}^\bullet_{\boldsymbol{\mathcal{P}}}(\boldsymbol{I}^{(r)}, \boldsymbol{I}^{(r)})$, uses at a key step the previous calculations of Friedlander and Suslin.
- Explicit representative for the extension classes $\boldsymbol{e}_1$ and $\boldsymbol{c}_1$,

$$0 \to \boldsymbol{I}_0^{(1)} \to \boldsymbol{S}^p \to \boldsymbol{\Gamma}^p \to \boldsymbol{I}_0^{(1)} \to 0$$

$$0 \to \boldsymbol{I}_0^{(1)} \to \boldsymbol{K}_p^0 \to \boldsymbol{K}_p^1 \to \cdots \to \boldsymbol{K}_p^{p-2} \to \boldsymbol{K}_p^{p-1} \to \boldsymbol{I}_1^{(1)} \to 0$$

  where $\boldsymbol{K}_p$ is the "super Koszul kernel subcomplex" of $\boldsymbol{\Omega}_p$.
- A byproduct of the arguments inspecting how $\boldsymbol{e}_1$ restricts to the Frobenius kernel $GL(m|n)_1$ is the following curious fact:

  For $m, n \geq 1$, one has $\mathrm{Ext}^2_{GL(m|n)}(k, k) \neq 0$.

Applications to support varieties . . . ? (in progress)